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On Domains of Attraction of Multivariate Extreme Value Distributions under Absolute Continuity

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The paper gives sufficient conditions for domains of attraction of multivariate extreme value distributions. Under the assumption of absolute continuity of a multivariate distribution, the criteria enable one to examine, by using limits of some rescaled conditional densities, whether the distribution belongs to the domain of

83–93], the criteria are easily applicable even when the marginal tails are not Pareto-like.

1. INTRODUCTION

Characterizing domains of attraction of univariate extreme value distributions is one of the classical problems in extreme value theory and was completely solved by Gnedenko [3]. There have been a number of attempts to extend this to the multivariate case. These include Berman [1], Rvaceva [14], de Haan and Resnick [4, 5, 7], Marshall and Olkin [12], de Haan and Omey [6], and de Haan *et al.* [8]. In particular, de Haan and Resnick [7] derived a criterion for domains of attraction of multivariate extreme value distributions in terms of densities. Their method is convenient to apply when marginal tails are Pareto-like. It is however clumsy for other cases, particularly when functional forms of marginal distribution functions cannot be explicitly obtained like gamma or normal distributions. In this paper we give sufficient conditions for domains of attraction of multivariate extreme value distributions which can be easily applied to most absolutely continuous multivariate distributions.

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Marshall and Olkin [12] obtained some multivariate analogs of Gnedenko's characterizations of domains of attraction. They handled every possible situation by building up different criteria according to the three different well-known types (Gumbel, Fréchet, and Weibull types) of univariate extreme value marginals. Since their criteria are however based on distribution functions, and since most multivariate distributions are specified by densities, not by distribution functions, the applications are limited.

On the other hand, de Haan and Resnick [7] obtained a criterion in terms of densities which implies regular variation of the tails of multivariate distributions. Regular variation is an efficient tool for describing domains of attraction of multivariate extreme value distributions when the univariate extreme value marginals are of Fréchet type. In other words, if there is given an absolutely continuous multivariate distribution whose marginal tails are Pareto-like, their criterion can be directly applied. If the marginal tails are not Pareto-like and if the functional forms of the marginal distribution functions cannot be explicitly obtained, it is of no use. Multivariate gamma and multivariate normal distributions are examples of this kind.

The purpose of this paper is to present general criteria for domains of attraction of multivariate extreme value distributions which cover this kind of example too. The criteria adopt a single parameter, called the shape parameter, to deal with all three different types of univariate extreme value marginals in a unified manner. Since a multivariate density can be expressed as a product of a univariate density and successive conditional densities, the criteria are developed in terms of these conditional densities instead of in terms of the multivariate density itself.

The rest of the paper is organized as follows. Section 2 provides technical preliminaries which are necessary to derive our criteria. Section 3 gives a criterion for a simple case where the limits of rescaled conditional densities are again proper densities. Section 4 extends this to more general cases. Section 5 contains four examples which show how our criteria are applied.

2. PRELIMINARIES

For $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \Re^k$ $(k \geqslant 1)$ and for $\alpha, \beta \in \Re$, we write

$$\mathbf{a}\mathbf{x} + \mathbf{b} = (a_1 x_1 + b_1, ..., a_k x_k + b_k),$$

$$\alpha\mathbf{x} + \beta = (\alpha x_1 + \beta, ..., \alpha x_k + \beta).$$

For a k-dimensional (k-dim.) distribution function F_k, if there exist k-dim. vectors $\mathbf{a}_n > \mathbf{0}$ (with componentwise ordering) and $\mathbf{b}_n \in \mathfrak{R}^k$, $n = 1, 2, ...$, such that

$$F_k^n(\mathbf{a}_n \mathbf{x} + \mathbf{b}_n) \xrightarrow{w} G_k(\mathbf{x}) \qquad \text{as} \quad n \to \infty \tag{2.1}$$

for some k-dim. nondegenerate distribution function G_k, where $\xrightarrow{w}$ denotes weak convergence, then G_k is called a k-dim. extreme value distribution and F_k is said to be in the domain of attraction of G_k, written $F_k \in \mathscr{D}(G_k)$. For a review on multivariate extreme value theory, see Galambos [2] and Resnick [13].

For $k = 1$, Gnedenko [3] showed that $F_1 \in \mathscr{D}(G_1)$ is equivalent to the condition that there exists a $\xi \in \mathfrak{R}$ such that

$$\lim_{u \uparrow x_{F_1}} \frac{1 - F_1(u + g(u)\,x)}{1 - F_1(u)} = (1 + \xi x)^{-1/\xi}, \qquad 1 + \xi x > 0, \tag{2.2}$$

where $x_{F_1} := \sup\{x\colon F_1(x) < 1\}$ and

$$\begin{aligned} &x_{F_1} = \infty \quad \text{and} \quad g(u) = \xi u && \text{if} \quad \xi > 0; \\ &g(u) \text{ is some strictly positive function} && \text{if} \quad \xi = 0; \\ &x_{F_1} < \infty \quad \text{and} \quad g(u) = -\xi(x_{F_1} - u) && \text{if} \quad \xi < 0. \end{aligned} \tag{2.3}$$

When $\xi = 0$, the function g is unique up to asymptotic equivalence (i.e., if there is another $\tilde{g}$ satisfying (2.2), then $\tilde{g}(u) \sim g(u)$ as $u \uparrow x_{F_1}$) and one appropriate choice of g is given by $g(u) = \int_u^{x_{F_1}} (1 - F_1(t))\, dt/(1 - F_1(u))$. If condition (2.2) holds for some $\xi \in \mathfrak{R}$, then

$$\begin{aligned} F_1^n(a_n x + b_n) \to \Omega_\xi(x) := \exp\{-(1 + \xi x)^{-1/\xi}\}, \\ 1 + \xi x > 0, \quad \text{as } n \to \infty, \end{aligned} \tag{2.4}$$

where $b_n = \inf\{x\colon F_1(x) \geqslant 1 - 1/n\}$ and $a_n = g(b_n)$, and therefore one may take $G_1 = \Omega_\xi$. Throughout this paper the case $\xi = 0$ is always interpreted as the limit $\xi \to 0$, i.e., $\Omega_0(x) = \exp(-e^{-x})$, $x \in \mathfrak{R}$, the standard Gumbel distribution. The Ω_ξ's for $\xi > 0$ and $\xi < 0$ are said to be of Fréchet and Weibull types, respectively. The ξ is called the shape parameter of univariate extreme value distributions. Condition (2.2) is in fact a reformulation of Theorem 1.6.2 of Leadbetter *et al.* [11].

Without loss of generality, the auxiliary function g in (2.3) is assumed to satisfy the following properties: as $u \uparrow x_{F_1}$,

$$
\begin{aligned}
&u+g(u)\,x \to x_{F_1} \qquad \text{for any } x \text{ with } 1+\xi x>0;\\
&g(u+g(u)\,x)/g(u) \to 1+\xi x\\
&\qquad\qquad \text{locally uniformly in } x \text{ with } 1+\xi x>0;\\
&(x_{F_1}-u)/g(u) \to 1/|\xi| \qquad \text{if} \quad x_{F_1}<\infty.
\end{aligned}
\tag{2.5}
$$

This is obviously true for $\xi \neq 0$. When $\xi = 0$, see Lemmas 1.2 and 1.3 and Proposition 1.4 of Resnick [13] for details which show that if (2.2) holds for some g which does not satisfy (2.5), then there exists another $\tilde{g}$ which satisfies (2.2) and (2.5), with $\tilde{g}(u) \sim g(u)$ as $u \uparrow x_{F_1}$.

If F_1 is absolutely continuous, then the well-known von Mises condition, a sufficient condition for $F_1 \in \mathscr{D}(G_1)$, is often more convenient to verify than condition (2.2). Specifically, if F_1 is absolutely continuous with probability density function f_1 and if there exists a $\xi \in \mathfrak{R}$ such that

$$
\lim_{u \uparrow x_{F_1}} \frac{g(u)\, f_1(u)}{1-F_1(u)} = 1, \tag{2.6}
$$

where x_{F_1} and g satisfy (2.3) and (2.5), then $F_1 \in \mathscr{D}(\Omega_\xi)$. This is a reformulation of Propositions 1.15–1.17 of Resnick [13]. In fact, the existence of f_1 in a left neighborhood of x_{F_1} is enough here since (2.6) is related only to the right tail of f_1. We need the following lemma to prove our main results in Sections 3 and 4.

LEMMA 2.1. *Let F_1 be a distribution function having a probability density function f_1. If there exists a $\xi \in \mathfrak{R}$ such that* (2.6) *holds, where x_{F_1} and g satisfy* (2.3) *and* (2.5), *then*

$$
\begin{aligned}
&\lim_{u \uparrow x_{F_1}} \frac{g(u)\, f_1(u+g(u)\,x)}{1-F_1(u)}\\
&\qquad = (1+\xi x)^{-1/\xi - 1} \qquad \textit{locally uniformly in } x \textit{ with } 1+\xi x>0.
\end{aligned}
$$

Proof. We prove continuous convergence. Let $x(u)$ be a function such that

$$
\lim_{u \uparrow x_{F_1}} x(u) = x, \qquad 1+\xi x(u)>0, \qquad 1+\xi x>0.
$$

Then, since (2.2) holds locally uniformly, it follows from this and (2.5) that

$$\begin{aligned}
&\frac{g(u)\, f_1(u+g(u)\, x(u))}{1-F_1(u)}\\
&\quad=\frac{g(u+g(u)\, x(u))\, f_1(u+g(u)\, x(u))}{1-F_1(u+g(u)\, x(u))}\cdot\frac{g(u)}{g(u+g(u)\, x(u))}\\
&\quad\quad\times\frac{1-F_1(u+g(u)\, x(u))}{1-F_1(u)}\\
&\quad\sim\frac{g(u)}{g(u+g(u)\, x(u))}\cdot\frac{1-F_1(u+g(u)\, x(u))}{1-F_1(u)}\\
&\quad\to(1+\xi x)^{-1/\xi-1} \qquad \text{as} \quad u\uparrow x_{F_1}.
\end{aligned}$$

This completes the proof. ∎

For $k \geqslant 2$, the G_k in (2.1) has no such finite-parameter representation as in (2.4) for G_1. A higher dimensional extension of (2.2) is however possible due to Marshall and Olkin [12]. For simplicity, F_k is assumed to have equal univariate marginals. That is, let $F_k(\mathbf{x})$ be a k-dim. distribution function with equal univariate marginals $F_1(x)$, and let $G_k(\mathbf{x})$ be a k-dim. extreme value distribution with equal univariate marginals $G_1(x)=\Omega_\xi(x)$ for some $\xi\in\mathfrak{R}$. Then $F_k\in\mathscr{D}(G_k)$ if and only if

$$\lim_{u\uparrow x_{F_1}}\frac{1-F_k(u+g(u)\,\mathbf{x})}{1-F_1(u)}=-\log G_k(\mathbf{x}), \qquad 1+\xi\mathbf{x}>\mathbf{0}, \tag{2.7}$$

where x_{F_1} and g satisfy (2.3) and (2.5). This result is a reformulation of Propositions 3.1–3.3 of Marshall and Olkin [12].

Unlike criterion (2.2) for univariate distributions, the G_k should first be specified to use criterion (2.7) for multivariate distributions. When we are given only a k-dim. distribution function F_k, the following lemma which removes the necessity of prior knowledge of G_k is more useful for checking $F_k\in\mathscr{D}(G_k)$ for some G_k and constructing such a G_k.

LEMMA 2.2. *Let $F_k(\mathbf{x})$ be a k-dim. ($k\geqslant 2$) distribution function with equal univariate marginals $F_1(x)$. If there exists a $\xi\in\mathfrak{R}$ such that*

$$V_k(\mathbf{x}):=\lim_{u\uparrow x_{F_1}}\frac{1-F_k(u+g(u)\,\mathbf{x})}{1-F_1(u)}, \qquad 1+\xi\mathbf{x}>\mathbf{0}, \tag{2.8}$$

exist finite and positive, where x_{F_1} and g satisfy (2.3) *and* (2.5), *then $F_k\in\mathscr{D}(G_k)$, where $G_k(\mathbf{x})=e^{-V_k(\mathbf{x})}$, $1+\xi\mathbf{x}>\mathbf{0}$, is a k-dim. extreme value* *distribution with equal univariate marginals* $G_1(x)=\Omega_\xi(x)$. *In this case,* V_k *necessarily satisfies*

$$V_k(t^\xi \mathbf{x}+(t^\xi-1)/\xi)=t^{-1}V_k(\mathbf{x}), \qquad 1+\xi\mathbf{x}>\mathbf{0}, \quad t>0. \tag{2.9}$$

Proof. Define $b_n=\inf\{x\colon F_1(x)\geqslant 1-1/n\}$ and $a_n=g(b_n)$. Then $n(1-F_1(b_n))\to 1$ as $n\to\infty$, and thus it follows from (2.8) with u replaced by b_n that

$$n(1-F_k(a_n\mathbf{x}+b_n))\to V_k(\mathbf{x}), \qquad 1+\xi\mathbf{x}>\mathbf{0}, \quad \text{as } n\to\infty.$$

But this is equivalent to

$$F_k^n(a_n\mathbf{x}+b_n)\to G_k(\mathbf{x})=e^{-V_k(\mathbf{x})}, \qquad 1+\xi\mathbf{x}>\mathbf{0}, \quad \text{as } n\to\infty.$$

By the Helly selection theorem and by (2.8), G_k must be a distribution function and so $F_k\in\mathscr{D}(G_k)$. Since this again implies $F_1\in\mathscr{D}(G_1)$, comparison of (2.2) with (2.8) leads to $G_1=\Omega_\xi$.

Now G_k is continuous since it is an extreme value distribution, and so (2.8) holds locally uniformly since monotone functions are converging to a continuous limit. Therefore, for $1+\xi\mathbf{x}>\mathbf{0}$ and $t>0$, writing $t_1=(t^\xi-1)/\xi$, we have from (2.2) and (2.5) that

$$\begin{aligned}
V_k(\mathbf{x}) &= \lim_{u\uparrow x_{F_1}} \frac{1-F_k(u+g(u)\,t_1+g(u+g(u)\,t_1)\,\mathbf{x})}{1-F_1(u+g(u)\,t_1)} \\
&= \lim_{u\uparrow x_{F_1}} \left[\frac{1-F_1(u)}{1-F_1(u+g(u)\,t_1)}\right. \\
&\qquad \left.\times\frac{1-F_k(u+g(u)(t_1+(g(u+g(u)\,t_1)/g(u))\,\mathbf{x}))}{1-F_1(u)}\right] \\
&= (1+\xi t_1)^{1/\xi}\, V_k(t_1+(1+\xi t_1)\,\mathbf{x}) \\
&= tV_k(t^\xi\mathbf{x}+(t^\xi-1)/\xi),
\end{aligned}$$

which completes the proof. ∎

Though this lemma is important in its own right, it still is not easy to apply this directly to F_k since most multivariate models for F_k are specified by densities, not by distribution functions. We now assume that F_k is an absolutely continuous, k-dim. ($k\geqslant 2$) distribution function with equal univariate marginals F_1. Let $(X_1, ..., X_k)$ be a random vector having distribution function F_k. We say F_k is *stationary* if $(X_1, ..., X_k)$ is part of a (strictly) stationary sequence of random variables. We use the following notation: for $i=1, ..., k$ we write

$$\mathbf{x}_i = (x_1, ..., x_i) \in \Re^i,$$
$$F_i(\mathbf{x}_i) = F_i(x_1, ..., x_i) = \mathbf{P}\{X_1 \leqslant x_1, ..., X_i \leqslant x_i\},$$
$$f_i(\mathbf{x}_i) = \partial^i F_i(\mathbf{x}_i)/(\partial x_1 \cdots \partial x_i),$$

and for $j=1, ..., k-1$ we write

$$\mathbf{x}_{j+1} = (x_1, ..., x_j, x_{j+1}) = (\mathbf{x}_j, x_{j+1}) \in \Re^{j+1},$$
$$f_{j+1}(x_{j+1} \mid \mathbf{x}_j) = f_{j+1}(\mathbf{x}_{j+1})/f_j(\mathbf{x}_j).$$

If it is really true that $F_k \in \mathscr{D}(G_k)$ for some k-dim. extreme value distribution G_k with equal univariate marginals $G_1 = \Omega_\xi$ for some $\xi \in \Re$, this also implies that $F_i \in \mathscr{D}(G_i)$, $i=1, ..., k$, where $G_i(\mathbf{x}_i) := G_k(\mathbf{x}_i, x_{G_1}, ..., x_{G_1})$, an i-dim. marginal of G_k, so that, for each $i=1, ..., k$, we have

$$\lim_{u \uparrow x_{F_1}} \frac{1-F_i(u+g(u)\,\mathbf{x}_i)}{1-F_1(u)} = -\log G_i(\mathbf{x}_i), \qquad 1+\xi\mathbf{x}_i > \mathbf{0},$$

from (2.2) and (2.7) with k replaced by i, where x_{F_1} and g satisfy (2.3) and (2.5). Taking partial derivatives in these convergences gives a plausibility of the existence of the limit

$$\begin{aligned} l_j(x_{j+1}\,;\mathbf{x}_j) := \lim_{u \uparrow x_{F_1}} g(u)\, f_{j+1}(u+g(u)\,x_{j+1} \mid u+g(u)\,\mathbf{x}_j), \\ 1+\xi\mathbf{x}_{j+1} > \mathbf{0}, \end{aligned} \tag{2.10}$$

for each $j=1, ..., k-1$. Based on these limits, sufficient conditions for $F_k \in \mathscr{D}(G_k)$ are given in Sections 3 and 4.

3. A SIMPLE CASE

Using the same notation as in Section 2, assume that, for each $j=1, ..., k-1$, the limit l_j in (2.10) exists finite. Though not true in general, there are many examples for which the limit $l_j(\,\cdot\,;\mathbf{x}_j)$ is a probability density function on $\{x: 1+\xi x > 0\}$ for every fixed $\mathbf{x}_j$ with $1+\xi\mathbf{x}_j > \mathbf{0}$. This section deals with the possibility of $F_k \in \mathscr{D}(G_k)$ for some multivariate extreme value distribution G_k under this simple case. Stationarity of F_k is assumed here. For the univariate marginals of F_k, we use the von Mises condition (2.6). Define $x^*_{F_1} := \inf\{x: F_1(x) > 0\}$.

THEOREM 3.1. *Let F_k be a k-dim. ($k \geqslant 2$) stationary distribution function having a probability density function f_k. Suppose that there exists a $\xi \in \Re$ such that (2.6) holds, and that, for each $j = 1, ..., k-1$, the limit l_j in (2.10) exists finite, where x_{F_1} and g satisfy (2.3) and (2.5). If, for each $j = 1, ..., k-1$, $l_j(\cdot\,; \mathbf{x}_j)$ is a probability density function on $\{x: 1+\xi x > 0\}$ for every fixed $\mathbf{x}_j$ with $1 + \xi \mathbf{x}_j > \mathbf{0}$, then $F_k \in \mathscr{D}(G_k)$ for some k-dim. extreme value distribution G_k with equal univariate marginals $G_1 = \Omega_\xi$; moreover, G_k is given so as to satisfy*

$$l_j(x_{j+1}\,; \mathbf{x}_j) = \left(\frac{\partial^{j+1} V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right), \qquad j = 1, ..., k-1, \tag{3.1}$$

provided that $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ is not zero, where $V_i(\mathbf{x}_i) = -\log G_i(\mathbf{x}_i)$, $i = 1, ..., k$.

Proof. We use the following notation:

$$f_1^{(u)}(x_1) := \frac{g(u)\, f_1(u + g(u)\, x_1)}{1 - F_1(u)}\,;$$

$$f_{j+1}^{(u)}(x_{j+1} \mid \mathbf{x}_j) := g(u)\, f_{j+1}(u + g(u)\, x_{j+1} \mid u + g(u)\, \mathbf{x}_j), \qquad 1 \leqslant j \leqslant k-1;$$

$$\psi_i^{(u)}(\mathbf{x}_i) := f_1^{(u)}(x_1) \left(\prod_{j=1}^{i-1} f_{j+1}^{(u)}(x_{j+1} \mid \mathbf{x}_j)\right), \qquad 2 \leqslant i \leqslant k;$$

$$\psi_i(\mathbf{x}_i) := (1 + \xi x_1)^{-1/\xi - 1} \left(\prod_{j=1}^{i-1} l_j(x_{j+1}\,; \mathbf{x}_j)\right), \qquad 2 \leqslant i \leqslant k.$$

Then, by Lemma 2.1, for each $i = 2, ..., k$, we have $\lim_{u \uparrow x_{F_1}} \psi_i^{(u)}(\mathbf{x}_i) = \psi_i(\mathbf{x}_i)$, $1 + \xi \mathbf{x}_i > \mathbf{0}$. We will first show that, for each $i = 2, ..., k$ and for every fixed x_1 with $1 + \xi x_1 > 0$,

$$\int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}} |\psi_i^{(u)}(\mathbf{y}_i) - \psi_i(\mathbf{y}_i)|\, dy_i \cdots dy_2\right) dy_i \to 0 \qquad \text{as} \quad u \uparrow x_{F_1}, \tag{3.2}$$

where $A_{i-1} := (x^*_{\Omega_\xi}, x_{\Omega_\xi})^{i-1} \subset \Re^{i-1}$. First of all, since every l_j is a probability density function on $(x^*_{\Omega_\xi}, x_{\Omega_\xi}) = \{y: 1 + \xi y > 0\}$, we have

$$\int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}} \psi_i(\mathbf{y}_i)\, dy_i \cdots dy_2\right) dy_1$$

$$= \int_{x_1}^{x_{\Omega_\xi}} (1 + \xi y_1)^{-1/\xi - 1}\, dy_1 = (1 + \xi x_1)^{-1/\xi} < \infty. \tag{3.3}$$

Also, since $(\psi_i - \psi_i^{(u)})_+ \leqslant \psi_i$ where x_+ means $\max\{x, 0\}$, we have, by the Lebesgue dominated convergence theorem,

$$\int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}} (\psi_i(\mathbf{y}_i) - \psi_i^{(u)}(\mathbf{y}_i))_+ \, dy_i \cdots dy_2 \right) dy_1 \to 0 \qquad \text{as} \quad u \uparrow x_{F_1}. \tag{3.4}$$

Next, using the Fatou lemma, observe that

$$\begin{aligned}
(1+\xi x_1)^{-1/\xi} &= \int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}} \psi_i(\mathbf{y}_i) \, dy_i \cdots dy_2 \right) dy_1 \\
&\leqslant \varliminf_{u \uparrow x_{F_1}} \int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}} \psi_i^{(u)}(\mathbf{y}_i) \, dy_i \cdots dy_2 \right) dy_1 \\
&\leqslant \varlimsup_{u \uparrow x_{F_1}} \int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}} \psi_i^{(u)}(\mathbf{y}_i) \, dy_i \cdots dy_2 \right) dy_1 \\
&\leqslant \varlimsup_{u \uparrow x_{F_1}} \int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{\Re^{i-1}} \psi_i^{(u)}(\mathbf{y}_i) \, dy_i \cdots dy_2 \right) dy_1 \\
&= \varlimsup_{u \uparrow x_{F_1}} \int_{x_1}^{x_{\Omega_\xi}} f_1^{(u)}(y_1) \, dy_1 \\
&= \lim_{u \uparrow x_{F_1}} \frac{1 - F_1(u + g(u)\, x_1)}{1 - F_1(u)} = (1+\xi x_1)^{-1/\xi},
\end{aligned}$$

where the last equality follows from (2.6) and thus from (2.2). In other words,

$$\begin{aligned}
&\int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}} \psi_i^{(u)}(\mathbf{y}_i) \, dy_i \cdots dy_2 \right) dy_1 \\
&\qquad \to (1+\xi x_1)^{-1/\xi} \qquad \text{as} \quad u \uparrow x_{F_1}.
\end{aligned} \tag{3.5}$$

Therefore, (3.2) follows immediately from (3.3), (3.4), and (3.5). In addition, it is also noted that, by (2.2) and (3.5),

$$\begin{aligned}
&\int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}^c} \psi_i^{(u)}(\mathbf{y}_i) \, dy_i \cdots dy_2 \right) dy_1 \\
&\quad = \frac{1 - F_1(u + g(u)\, x_1)}{1 - F_1(u)} - \int_{x_1}^{x_{\Omega_\xi}} \left(\int \cdots \int_{A_{i-1}} \psi_i^{(u)}(\mathbf{y}_i) \, dy_i \cdots dy_2 \right) dy_1 \\
&\quad \to (1+\xi x_1)^{-1/\xi} - (1+\xi x_1)^{-1/\xi} = 0 \qquad \text{as} \quad u \uparrow x_{F_1},
\end{aligned} \tag{3.6}$$

where $A_{i-1}^c := \Re^{i-1} \backslash A_{i-1}$.

We now show the existence of the limit of $(1 - F_k(u + g(u)\, \mathbf{x}_k))/(1 - F_1(u))$, $1 + \xi \mathbf{x}_k > \mathbf{0}$, as $u \uparrow x_{F_1}$. If we let $(X_1, ..., X_k) \sim F_k$, then

$$
\begin{aligned}
1-F_k(u+g(u)\,\mathbf{x}_k)
&=1-\mathbf{P}\{X_1\leqslant u+g(u)\,x_1,\,...,\,X_k\leqslant u+g(u)\,x_k\}\\
&=\sum_{i=1}^{k}\mathbf{P}\{X_i>u+g(u)\,x_i\}\\
&\quad+\sum_{\substack{D\subset\{1,\,...,\,k\}\\|D|\geqslant 2}}(-1)^{|D|-1}\,\mathbf{P}\{X_i>u+g(u)\,x_i,\,i\in D\}.
\end{aligned}\tag{3.7}
$$

Here, for $D=\{i_1<\cdots<i_r\}\subset\{1,\,...,\,k\}$ with $r\geqslant 2$,

$$
\begin{aligned}
&\mathbf{P}\{X_i>u+g(u)\,x_i,\,i\in D\}\\
&\quad=\mathbf{P}\{X_{i_j}>u+g(u)\,x_{i_j},\,j=1,\,...,\,r\}\\
&\quad=\mathbf{P}\{X_{i_j-i_1+1}>u+g(u)\,x_{i_j},\,j=1,\,...,\,r\}\qquad\text{(by the stationarity of } F_k\text{)}\\
&\quad=\int_{x_{i_1}}^{\infty}\left(\int\cdots\int_{B_D}(g(u))^{i_r-i_1+1}\right.\\
&\qquad\left.\times f_{i_r-i_1+1}(u+g(u)\,\mathbf{y}_{i_r-i_1+1})\,dy_{i_r-i_1+1}\cdots dy_2\right)dy_1,
\end{aligned}
$$

where $B_D=\{(z_1,\,...,\,z_{i_r-i_1})\in\Re^{i_r-i_1}\colon z_{i_j-i_1}>x_{i_j},\,j=2,\,...,\,r\}$, and so

$$
\begin{aligned}
&\frac{\mathbf{P}\{X_i>u+g(u)\,x_i,\,i\in D\}}{1-F_1(u)}\\
&\quad=\int_{x_{i_1}}^{\infty}\left(\int\cdots\int_{B_D}\psi^{(u)}_{i_r-i_1+1}(\mathbf{y}_{i_r-i_1+1})\,dy_{i_r-i_1+1}\cdots dy_2\right)dy_1\\
&\quad=\int_{x_{i_1}}^{x_{\Omega_\xi}}\left(\int\cdots\int_{B_D\cap A_{i_r-i_1}}\psi^{(u)}_{i_r-i_1+1}(\mathbf{y}_{i_r-i_1+1})\,dy_{i_r-i_1+1}\cdots dy_2\right)dy_1\\
&\qquad+\int_{x_{i_1}}^{x_{\Omega_\xi}}\left(\int\cdots\int_{B_D\cap A^c_{i_r-i_1}}\psi^{(u)}_{i_r-i_1+1}(\mathbf{y}_{i_r-i_1+1})\,dy_{i_r-i_1+1}\cdots dy_2\right)dy_1\\
&\quad\to\int_{x_{i_1}}^{x_{\Omega_\xi}}\left(\int\cdots\int_{B_D\cap A_{i_r-i_1}}\psi_{i_r-i_1+1}(\mathbf{y}_{i_r-i_1+1})\,dy_{i_r-i_1+1}\cdots dy_2\right)dy_1\\
&\qquad\qquad\text{(by (3.2) and (3.6))}\\
&\quad=\int_{x_{i_1}}^{x_{\Omega_\xi}}\int_{x^*_{\Omega_\xi}}^{x_{\Omega_\xi}}\cdots\int_{x^*_{\Omega_\xi}}^{x_{\Omega_\xi}}\int_{x_{i_2}}^{x_{\Omega_\xi}}\int_{x^*_{\Omega_\xi}}^{x_{\Omega_\xi}}\cdots\int_{x^*_{\Omega_\xi}}^{x_{\Omega_\xi}}\int_{x_{i_r}}^{x_{\Omega_\xi}}\psi_{i_r-i_1+1}(\mathbf{y}_{i_r-i_1+1})\,d\mathbf{y}_{i_r-i_1+1}\\
&\quad\equiv J_k(D;\mathbf{x}_k)\qquad\text{as}\quad u\uparrow x_{F_1}.
\end{aligned}\tag{3.8}
$$

Together with (2.2), it therefore follows that

$$\frac{1-F_k(u+g(u)\,\mathbf{x}_k)}{1-F_1(u)} \to \sum_{i=1}^{k} (1+\xi x_i)^{-1/\xi} + \sum_{\substack{D\subset\{1,\,...,\,k\}\\ |D|\geqslant 2}} (-1)^{|D|-1} J_k(D;\mathbf{x}_k)$$

$$\equiv V_k(\mathbf{x}_k), \qquad 1+\xi\,\mathbf{x}_k>\mathbf{0}, \qquad \text{as} \quad u\uparrow x_{F_1}. \tag{3.9}$$

Hence, by Lemma 2.2, $F_k \in \mathscr{D}(G_k)$, where $G_k(\mathbf{x}_k)=e^{-V_k(\mathbf{x}_k)}$, $1+\xi\,\mathbf{x}_k>\mathbf{0}$. The last assertion (3.1) is immediate since

$$l_j(x_{j+1}\,;\mathbf{x}_j)=\frac{\psi_{j+1}(\mathbf{x}_{j+1})}{\psi_j(\mathbf{x}_j)}=\left(\frac{\partial^{j+1}V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1\cdots\partial x_{j+1}}\right)\Big/\left(\frac{\partial^{j}V_{j}(\mathbf{x}_{j})}{\partial x_1\cdots\partial x_{j}}\right).$$

The proof is completed. ∎

Remark 3.2. Relations (2.9) and (3.1) imply that, for each $j=1,\,...,\,k-1$, $(1+\xi x_{j+1})\,l_j(x_{j+1}\,;\mathbf{x}_j)$ is necessarily a function of

$$\nabla\mathbf{x}_{j+1}:=\left(\frac{1}{\xi}\log\left(\frac{1+\xi x_2}{1+\xi x_1}\right),\,...,\,\frac{1}{\xi}\log\left(\frac{1+\xi x_{j+1}}{1+\xi x_j}\right)\right),$$

where $\nabla\mathbf{x}_{j+1}$ is interpreted as $(x_2-x_1,\,...,\,x_{j+1}-x_j)$ when $\xi=0$. To see this, we apply $t=(1+\xi x_{j+1})^{-1/\xi}$ to (2.9) with k replaced by $j+1$, and get

$$V_{j+1}(\mathbf{x}_{j+1})=(1+\xi x_{j+1})^{-1/\xi}\,V_{j+1}((1+\xi x_{j+1})^{-1}\,(\mathbf{x}_j-x_{j+1}),0),$$
$$1+\xi\,\mathbf{x}_{j+1}>\mathbf{0}.$$

Similarly, we get

$$V_j(\mathbf{x}_j)=(1+\xi x_{j+1})^{-1/\xi}\,V_j((1+\xi x_{j+1})^{-1}\,(\mathbf{x}_j-x_{j+1})), \qquad 1+\xi\,\mathbf{x}_{j+1}>\mathbf{0}.$$

Then, writing $z_i=(1+\xi x_{j+1})^{-1}\,(x_i-x_{j+1})$, $i=1,\,...,\,j$, we have

$$\frac{\partial^{j}V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1\cdots\partial x_j}=(1+\xi x_{j+1})^{-1/\xi-j}\,\frac{\partial^{j}V_{j+1}(\mathbf{z}_j,0)}{\partial z_1\cdots\partial z_j},$$

$$\frac{\partial^{j}V_{j}(\mathbf{x}_{j})}{\partial x_1\cdots\partial x_j}=(1+\xi x_{j+1})^{-1/\xi-j}\,\frac{\partial^{j}V_{j}(\mathbf{z}_j)}{\partial z_1\cdots\partial z_j}.$$

Therefore, their ratio is a function of $(1+\xi x_{j+1})^{-1}\,(\mathbf{x}_j-x_{j+1})$ and thus a function of $\nabla\mathbf{x}_{j+1}$, and so we may write

$$H_j\left(\frac{1}{\xi}\log\left(\frac{1+\xi x_{j+1}}{1+\xi x_j}\right);\nabla\mathbf{x}_j\right)=\left(\frac{\partial^{j}V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1\cdots\partial x_j}\right)\Big/\left(\frac{\partial^{j}V_{j}(\mathbf{x}_{j})}{\partial x_1\cdots\partial x_j}\right).$$

Finally, writing $h_j(y; \nabla \mathbf{x}_j) = \partial H_j(y; \nabla \mathbf{x}_j)/\partial y$, we have

$$l_j(x_{j+1}; \mathbf{x}_j) = \left(\frac{\partial^{j+1} V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right)$$
$$= \frac{1}{1+\xi x_{j+1}} h_j\left(\frac{1}{\xi} \log\left(\frac{1+\xi x_{j+1}}{1+\xi x_j}\right); \nabla \mathbf{x}_j\right),$$

which implies that $(1+\xi x_{j+1})\, l_j(x_{j+1}; \mathbf{x}_j)$ is a function of $\nabla \mathbf{x}_{j+1}$.

4. EXTENSIONS TO MORE GENERAL CASES

As pointed out in Section 3, the limit $l_j(\,\cdot\,; \mathbf{x}_j)$ in (2.10) need not be in general a probability density function on $\{x: 1+\xi x > 0\}$. This section deals with this general case. We work with the concept of locally uniform integrability of a class of functions. A class of real-valued functions defined on $\Re$ is said to be *locally uniformly integrable* over an unbounded interval if the class is uniformly integrable over any compact subset of that interval. For example, real-valued functions defined on $\Re$ are locally uniformly integrable if they are dominated by a continuous function. To simplify the proof, the case of symmetry of F_k is handled first, and then the non-symmetric case follows. In the non-symmetric case, the idea is basically the same, but notation and formulation are much more complicated.

THEOREM 4.1. *Let F_k be a k-dim. $(k \geqslant 2)$ symmetric distribution function having a probability density function f_k. Suppose that there exists a $\xi \in \Re$ such that* (2.6) *holds, and that, for each $j = 1, ..., k-1$, the limit l_j in* (2.10) *exists finite, where x_{F_1} and g satisfy* (2.3) *and* (2.5). *If, for each $j = 1, ..., k-1$ and for every fixed $\mathbf{x}_j$ with $1+\xi \mathbf{x}_j > \mathbf{0}$, there exists a $u_j^*(\mathbf{x}_j) < x_{F_1}$ such that the class*

$$\{g(u)\, f_{j+1}(u + g(u)\, x_{j+1} \mid u + g(u)\, \mathbf{x}_j) : u_j^*(\mathbf{x}_j) \leqslant u < x_{F_1}\}$$

of functions of x_{j+1} is locally uniformly integrable over $\{x_{j+1}: 1+\xi x_{j+1} > 0\}$, then $F_k \in \mathscr{D}(G_k)$ for some k-dim. extreme value distribution G_k with equal univariate marginals $G_1 = \Omega_\xi$; moreover, G_k is given so as to satisfy (3.1), *provided that $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ is not zero, where $V_i(\mathbf{x}_i) = -\log G_i(\mathbf{x}_i)$, $i = 1, ..., k$.*

Proof. Let $(X_1, ..., X_k) \sim F_k$. Then, at relation (3.7) with $1+\xi \mathbf{x}_k > \mathbf{0}$, since, for $D = \{i_1 < \cdots < i_r\} \subset \{1, ..., k\}$ with $r \geqslant 2$,

$$\mathbf{P}\{X_i > u + g(u)\, x_i,\, i \in D\} = \mathbf{P}\{X_{i_j} > u + g(u)\, x_{i_j},\, j = 1, ..., r\}$$
$$= \mathbf{P}\{X_j > u + g(u)\, x_{i_j},\, j = 1, ..., r\}$$

by the symmetry of F_k, we will show that

$$\lim_{u \uparrow x_{F_1}} \frac{\mathbf{P}\{X_j > u + g(u)\, x_{i_j},\ j = 1, ..., r\}}{1 - F_1(u)} = \int_{x_{i_1}}^{x_{\Omega_\xi}} \cdots \int_{x_{i_r}}^{x_{\Omega_\xi}} \psi_r(\mathbf{y}_r)\, d\mathbf{y}_r$$
$$\equiv J_k(D; \mathbf{x}_k), \tag{4.1}$$

where ψ_r is the function defined in the proof of Theorem 3.1, and conclude that

$$\lim_{u \uparrow x_{F_1}} \frac{1 - F_k(u + g(u)\, \mathbf{x}_k)}{1 - F_1(u)}$$
$$= \sum_{i=1}^{k} (1 + \xi x_i)^{-1/\xi} + \sum_{\substack{D \subset \{1, ..., k\} \\ |D| \geqslant 2}} (-1)^{|D| - 1} J_k(D; \mathbf{x}_k)$$
$$\equiv V_k(\mathbf{x}_k), \qquad 1 + \xi \mathbf{x}_k > \mathbf{0}. \tag{4.2}$$

For $M > 1$, define $b_\xi(M, x) = ((1 + \xi x) M^\xi - 1)/\xi$. Then $b_\xi(M, x) > x$ and $1 + \xi b_\xi(M, x) > 0$ for any x with $1 + \xi x > 0$. Also, by (2.6) and thus by (2.2), for $j = 1, ..., r$,

$$\lim_{M \to \infty} \lim_{u \uparrow x_{F_1}} \frac{\mathbf{P}\{X_j > u + g(u)\, b_\xi(M, x_{i_j})\}}{1 - F_1(u)} = \lim_{M \to \infty} M^{-1} (1 + \xi x_{i_j})^{-1/\xi} = 0,$$

and so we have

$$\lim_{u \uparrow x_{F_1}} \frac{\mathbf{P}\{X_j > u + g(u)\, x_{i_j},\ j = 1, ..., r\}}{1 - F_1(u)}$$
$$= \lim_{M \to \infty} \lim_{u \uparrow x_{F_1}} \frac{\mathbf{P}\{u + g(u)\, x_{i_j} < X_j \leqslant u + g(u)\, b_\xi(M, x_{i_j}),\ j = 1, ..., r\}}{1 - F_1(u)}.$$

Here, using the same notation as in the proof of Theorem 3.1, it can be seen that

$$\frac{\mathbf{P}\{u + g(u)\, x_{i_j} < X_j \leqslant u + g(u)\, b_\xi(M, x_{i_j}),\ j = 1, ..., r\}}{1 - F_1(u)}$$
$$= \int_{x_{i_1}}^{b_\xi(M, x_{i_1})} \cdots \int_{x_{i_r}}^{b_\xi(M, x_{i_r})} \psi_r^{(u)}(\mathbf{y}_r)\, dy_r \cdots dy_1$$
$$\to \int_{x_{i_1}}^{b_\xi(M, x_{i_1})} \cdots \int_{x_{i_r}}^{b_\xi(M, x_{i_r})} \psi_r(\mathbf{y}_r)\, dy_r \cdots dy_1 \qquad \text{as} \quad u \uparrow x_{F_1},$$

by applying successively the locally uniform integrability of $f^{(u)}_{j+1}(\,\cdot \mid \mathbf{y}_j)$ and then by applying Lemma 2.1. Finally, letting $M \to \infty$ leads to (4.1) since $b_\xi(M, x_{i_j}) \uparrow x_{\Omega_\xi}$ as $M \to \infty$ for $j = 1, ..., r$. Therefore, from (4.2) and thus from Lemma 2.2, $F_k \in \mathscr{D}(G_k)$, where $G_k(\mathbf{x}_k) = e^{-V_k(\mathbf{x}_k)}$, $1 + \xi \mathbf{x}_k > \mathbf{0}$. The last assertion (3.1) is immediate. This completes the proof. ▮

When F_k is not symmetric, the conditional densities in Theorem 4.1 are not enough to get a criterion for domains of attraction. In this case, we need more conditional densities which can explain the distribution of every subset of $\{X_1, ..., X_k\}$, where $(X_1, ..., X_k) \sim F_k$. For this purpose, we adopt new notation as: for any $D = \{i_1 < \cdots < i_r\} \subset \{1, ..., k\}$ with $r \geqslant 2$, let $f_{i_r \mid i_1, ..., i_{r-1}}(x_{i_r} \mid x_{i_1}, ..., x_{i_{r-1}})$ denote the conditional density of X_{i_r} given $(X_{i_1}, ..., X_{i_{r-1}}) = (x_{i_1}, ..., x_{i_{r-1}})$.

THEOREM 4.2. *Let F_k be a k-dim. ($k \geqslant 2$) distribution function with equal univariate marginals F_1, having a probability density function f_k. Suppose that there exists a $\xi \in \mathfrak{R}$ such that* (2.6) *holds, where x_{F_1} and g satisfy* (2.3) *and* (2.5). *If, for any $D = \{i_1 < \cdots < i_r\} \subset \{1, ..., k\}$ with $r \geqslant 2$,*

$$l_{i_r \mid i_1, ..., i_{r-1}}(x_{i_r}; x_{i_1}, ..., x_{i_{r-1}}) = \lim_{u \uparrow x_{F_1}} g(u)\, f_{i_r \mid i_1, ..., i_{r-1}}(u + g(u)\, x_{i_r} \mid u + g(u)\, x_{i_1}, ..., u + g(u)\, x_{i_{r-1}}),$$

$$1 + \xi x_{i_j} > 0,$$

exist finite and if, in addition, for every fixed $x_{i_1}, ..., x_{i_{r-1}}$ with $1 + \xi x_{i_j} > 0$, there exists a $u^*(x_{i_1}, ..., x_{i_{r-1}}) < x_{F_1}$ such that the class

$$\{g(u)\, f_{i_r \mid i_1, ..., i_{r-1}}(u + g(u)\, x_{i_r} \mid u + g(u)\, x_{i_1}, ..., u + g(u)\, x_{i_{r-1}}): u^*(x_{i_1}, ..., x_{i_{r-1}}) \leqslant u < x_{F_1}\}$$

of functions of x_{i_r} is locally uniformly integrable over $\{x_{i_r}: 1 + \xi x_{i_r} > 0\}$, then $F_k \in \mathscr{D}(G_k)$ for some k-dim. extreme value distribution G_k with equal univariate marginals $G_1 = \Omega_\xi$.

Proof. The proof is basically similar to that of Theorem 4.1 and so is omitted. ▮

5. EXAMPLES

In this section we consider four examples to show how our criteria in Sections 3 and 4 are applied. It is noted that the marginal tails in Examples 5.1, 5.3, and 5.4 are not Pareto-like.

EXAMPLE 5.1 (Multivariate Exponential Distribution). A multivariate exponential distribution F_k with parameter $\alpha > 0$ is defined by its density f_k as (see Johnson and Kotz [10, page 288]):

$$f_k(\mathbf{x}_k) = \left\{ \prod_{s=1}^{k-1} \left(1 + \frac{s}{\alpha}\right) \right\} \left(\sum_{s=1}^{k} e^{x_s/\alpha} - k + 1 \right)^{-(\alpha+k)} \times \exp\left(\frac{1}{\alpha} \sum_{s=1}^{k} x_s \right), \qquad \mathbf{x}_k > \mathbf{0}. \tag{5.1}$$

Here, the univariate marginal distribution F_1 is the standard exponential distribution so that (2.6) holds with $\xi = 0$ and $g(u) = 1$ and thus $F_1 \in \mathscr{D}(\Omega_0)$. Note also that every lower dimensional marginal density function $f_i(\mathbf{x}_i)$ $(i = 1, ..., k-1)$ is of the same form as (5.1). Therefore, it can be easily shown that, for each $j = 1, ..., k-1$,

$$\begin{aligned} l_j(x_{j+1}; \mathbf{x}_j) &= \lim_{u \to \infty} f_{j+1}(u + x_{j+1} \mid u + \mathbf{x}_j) \\ &= \left(1 + \frac{j}{\alpha}\right) \left(\frac{e^{x_{j+1}/\alpha}}{\sum_{s=1}^{j} e^{x_s/\alpha}} \right) \left(1 + \frac{e^{x_{j+1}/\alpha}}{\sum_{s=1}^{j} e^{x_s/\alpha}} \right)^{-(j+\alpha+1)}, \\ & \quad \mathbf{x}_{j+1} \in \mathfrak{R}^{j+1}. \end{aligned}$$

Note here that each $l_j(x; \mathbf{x}_j)$ is the density function of the random variable $X = \alpha\{\log(\sum_{s=1}^{j} e^{x_s/\alpha}) - \log(1/Z - 1)\}$, where $Z \sim \text{Beta}(1, j + \alpha)$. Thus, by Theorem 3.1, $F_k \in \mathscr{D}(G_k)$ for some k-dim. extreme value distribution G_k with equal univariate marginals $G_1 = \Omega_0$. We now construct the explicit form of G_k using (3.8) and (3.9). Since

$$\psi_i(\mathbf{y}_i) = e^{-y_1} \left(\prod_{j=1}^{i-1} l_j(y_{j+1}; \mathbf{y}_j) \right) = \left\{ \prod_{s=1}^{i-1} \left(1 + \frac{s}{\alpha}\right) \right\} \frac{\exp(\alpha^{-1} \sum_{s=1}^{i} y_s)}{(\sum_{s=1}^{i} e^{y_s/\alpha})^{i+\alpha}},$$

we can easily show that $J_k(D; \mathbf{x}_k) = (\sum_{i \in D} e^{x_i/\alpha})^{-\alpha}$ for any subset D of $\{1, ..., k\}$ with $|D| \geqslant 2$. Hence, it follows from (3.9) that $F_k \in \mathscr{D}(G_k)$, where

$$-\log G_k(\mathbf{x}_k) = \sum_{\substack{D \subset \{1, ..., k\} \\ D \neq \varnothing}} (-1)^{|D|-1} \left(\sum_{i \in D} e^{x_i/\alpha} \right)^{-\alpha}, \qquad \mathbf{x}_k \in \mathfrak{R}^k.$$

This multivariate extreme value distribution was, in fact, introduced recently by Joe [9].

EXAMPLE 5.2 (Multivariate t-Distribution). A multivariate t-distribution F_k with parameter v (v is a positive integer) is defined by its density f_k as (see Johnson and Kotz [10, page 134]):

$$f_k(\mathbf{x}_k) = \frac{\Gamma((v+k)/2)}{(\pi v)^{k/2}\,\Gamma(v/2)} \left(1 + v^{-1} \sum_{s=1}^{k} x_s^2\right)^{-(v+k)/2}, \qquad \mathbf{x}_k \in \mathfrak{R}^k. \quad (5.2)$$

Here, the univariate marginal distribution F_1 is the t-distribution with degree of freedom v so that (2.6) holds with $\xi = 1/v$ and $g(u) = v^{-1}u$ and thus $F_1 \in \mathscr{D}(\Omega_{1/v})$. Since every lower dimensional marginal density function $f_i(\mathbf{x}_i)$ ($i = 1, ..., k-1$) is of the same form as (5.2), we can easily show that, for each $j = 1, ..., k-1$,

$$\begin{aligned} &l_j(x_{j+1}\,;\mathbf{x}_j) \\ &\quad = \lim_{u\to\infty} v^{-1}uf_{j+1}(u + v^{-1}ux_{j+1} \mid u + v^{-1}u\mathbf{x}_j) \\ &\quad = \frac{\Gamma((v+j+1)/2)}{v\sqrt{\pi}\,\Gamma((v+j)/2)} \left\{\sum_{s=1}^{j} (1 + v^{-1}x_s)^2\right\}^{-1/2} \\ &\qquad \times \left\{1 + \frac{(1 + v^{-1}x_{j+1})^2}{\sum_{s=1}^{j} (1 + v^{-1}x_s)^2}\right\}^{-(v+j+1)/2}, \qquad x_1, ..., x_{j+1} > -v. \end{aligned}$$

Note here that $2l_j(x;\mathbf{x}_j)$ is the density function of the random variable $X = \{\sum_{s=1}^{j} (x_s + v)^2/(1/Z - 1)\}^{1/2} - v$, where $Z \sim \text{Beta}(1/2, (v+j)/2)$. In other words, $l_j(x;\mathbf{x}_j)$ gives a total mass 1/2 on $x > -v$, and so Theorem 3.1 cannot be applied. The symmetry of the functional form of (5.2) instead indicates possibility of applying Theorem 4.1. In fact, for each $j = 1, ..., k-1$ and for every fixed $x_1, ..., x_j > -v$, since

$$\begin{aligned} &v^{-1}uf_{j+1}(u + v^{-1}ux_{j+1} \mid u + v^{-1}u\mathbf{x}_j) \\ &\quad = \frac{\Gamma((v+j+1)/2)}{\sqrt{\pi}\,\Gamma((v+j)/2)} \cdot \frac{u\{1 + v^{-1}u^2 \sum_{s=1}^{j+1} (1 + v^{-1}x_s)^2\}^{-(v+j+1)/2}}{v\sqrt{v}\,\{1 + v^{-1}u^2 \sum_{s=1}^{j} (1 + v^{-1}x_s)^2\}^{-(v+j)/2}} \\ &\quad \leqslant \frac{v^{(v+j)/2-1}\Gamma((v+j+1)/2)}{\sqrt{\pi}\,\Gamma((v+j)/2)} \\ &\qquad \times \frac{\{(u^*)^{-2} + v^{-1} \sum_{s=1}^{j} (1 + v^{-1}x_s)^2\}^{(v+j)/2}}{\{\sum_{s=1}^{j} (1 + v^{-1}x_s)^2\}^{1/2} (1 + v^{-1}x_{j+1})^{v+j}}, \\ &\qquad x_{j+1} > -v, \qquad u \geqslant u^*, \end{aligned}$$

for any fixed $u^* > 0$, where the last one is a continuous function of x_{j+1}, the rescaled conditional densities are locally uniformly integrable over

$\{x_{j+1}: x_{j+1} > -v\}$. Thus, by Theorem 4.1, $F_k \in \mathscr{D}(G_k)$ for some k-dim. extreme value distribution G_k with equal univariate marginals $G_1 = \Omega_{1/v}$. In fact, from (4.1) and (4.2), G_k is given by

$$-\log G_k(\mathbf{x}_k) = \sum_{i=1}^{k} (1+v^{-1}x_i)^{-v} + \sum_{\substack{D \subset \{1, ..., k\} \\ |D| \geqslant 2}} (-1)^{|D|-1} J_k(D; \mathbf{x}_k),$$

$$1 + v^{-1}\mathbf{x}_k > \mathbf{0}, \tag{5.3}$$

where

$$J_k(D; \mathbf{x}_k) = \frac{\Gamma((v+r)/2)}{(v\sqrt{\pi})^{r-1}\,\Gamma((v+1)/2)} \int_{x_{i_1}}^{\infty} \cdots \int_{x_{i_r}}^{\infty} \left\{ \sum_{s=1}^{r} (1+v^{-1}y_s)^2 \right\}^{-(v+r)/2} d\mathbf{y}_r$$

for $D = \{i_1 < \cdots < i_r\} \subset \{1, ..., k\}$ with $r \geqslant 2$. In particular, when $k=2$ and $v=1$, which corresponds to the bivariate Cauchy distribution, (5.3) is reduced to

$$-\log G_2(x_1, x_2) = (1/2)[(1+x_1)^{-1} + (1+x_2)^{-1} + \{(1+x_1)^{-2} + (1+x_2)^{-2}\}^{1/2}],$$

where $x_1, x_2 > -1$. Compare this with Example 5.19 of Resnick [13].

EXAMPLE 5.3 (Multivariate Gamma Distribution). A multivariate gamma distribution F_k with parameter $\alpha > 0$ is defined by its density f_k as (see Johnson and Kotz [10, p. 217]):

$$f_k(\mathbf{x}_k) = \frac{1}{\Gamma(\alpha)} \left\{ \exp\left(-\sum_{s=1}^{k} x_s \right) \right\} \int_0^{\tilde{x}_k} t^{\alpha-1} e^{(k-1)t}\, dt, \qquad \mathbf{x}_k > \mathbf{0}, \tag{5.4}$$

where $\tilde{x}_k := \min\{x_1, ..., x_k\}$. Here, the univariate marginal distribution F_1 is the standard gamma distribution with parameter $\alpha+1$, i.e., $f_1(x_1) = (\Gamma(\alpha+1))^{-1} x_1^{\alpha} e^{-x_1}$, $x_1 > 0$, so that (2.6) holds with $\xi = 0$ and $g(u) = 1$ and thus $F_1 \in \mathscr{D}(\Omega_0)$. Note also that every lower dimensional marginal density function $f_i(\mathbf{x}_i)$ $(i = 1, ..., k-1)$ is of the same form as (5.4). Therefore, it can be easily shown that, for each $j = 1, ..., k-1$,

$$l_j(x_{j+1}; \mathbf{x}_j) = \lim_{u \to \infty} f_{j+1}(u + x_{j+1} \mid u + \mathbf{x}_j) = (1 - 1/j)\, e^{j\tilde{x}_{j+1} - x_{j+1} - (j-1)\tilde{x}_j}, \qquad \mathbf{x}_{j+1} \in \mathfrak{R}^{j+1}.$$

Here, observe that, for each $j=2, ..., k-1$, the limit $l_j(\cdot\,; \mathbf{x}_j)$ is a probability density function on $\mathfrak{R}$, whereas $l_1(\cdot; x_1)=0$ is not. Also, observe that, for each $j=1, ..., k-1$ and for every fixed $\mathbf{x}_j \in \mathfrak{R}^j$,

$$f_{j+1}(u+x_{j+1} \mid u+\mathbf{x}_j) = \frac{e^{-u-x_{j+1}} \int_0^{u+\tilde{x}_{j+1}} t^{\alpha-1} e^{jt}\, dt}{\int_0^{u+\tilde{x}_j} t^{\alpha-1} e^{(j-1)t}\, dt}$$
$$\leqslant e^{\tilde{x}_{j+1}-x_{j+1}}, \qquad x_{j+1} \in \mathfrak{R}, \quad u \geqslant u_j^*(\mathbf{x}_j),$$

where $u_j^*(\mathbf{x}_j) = \max\{-x_1, ..., -x_j, 0\} + 1$. Hence, by Theorem 4.1, $F_k \in \mathscr{D}(G_k)$, where G_k, from (4.1) and (4.2), is given by

$$G_k(\mathbf{x}_k) = \exp\left(-\sum_{i=1}^{k} e^{-x_i}\right), \qquad \mathbf{x}_k \in \mathfrak{R}^k. \tag{5.5}$$

EXAMPLE 5.4 (Multivariate Normal Distribution). Let F_k be the k-dim. normal distribution with equal univariate marginals $F_1 \sim N(0, 1)$. It is well known that if all correlations are absolutely less than 1, then $F_k \in \mathscr{D}(G_k)$, where G_k is given by (5.5). We show this by using our method. According to Proposition 5.27 of Resnick [13], it is enough to show this for $k=2$. We write the density f_2 of F_2 as:

$$f_2(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right\}, \qquad x_1, x_2 \in \mathfrak{R} \quad (|\rho|<1).$$

Since (2.6) holds with $\xi=0$ and $g(u)=1/u$, $F_1 \in \mathscr{D}(\Omega_0)$ and moreover

$$l_1(x_2\,; x_1) = \lim_{u\to\infty} u^{-1} f_2(u+u^{-1}x_2 \mid u+u^{-1}x_1) = 0, \qquad x_1, x_2 \in \mathfrak{R}.$$

Also, observe that, for every fixed $x_1 \in \mathfrak{R}$,

$$u^{-1} f_2(u+u^{-1}x_2 \mid u+u^{-1}x_1) \leqslant \frac{1}{u^*\sqrt{2\pi(1-\rho^2)}}, \qquad x_2 \in \mathfrak{R}, \quad u \geqslant u^*,$$

for any fixed $u^*>0$. Therefore, by Theorem 4.1, $F_2 \in \mathscr{D}(G_2)$, where G_2, from (4.1) and (4.2), is given by $G_2(x_1, x_2) = \exp(-e^{-x_1} - e^{-x_2})$, $x_1, x_2 \in \mathfrak{R}$.

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